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Factorization of the two loop four-particle amplitude in superstring theory revisited

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Abstract: We study in detail the factorization of the newly obtained two-loop four-particle amplitude in superstring theory. In particular some missing factors from the scalar correlators are obtained correctly, in comparing with a previous study of the factorization in two-loop superstring theory. Some details for the calculation of the factorization of the kinematic factor are also presented.

Keywords: Field Theories in Higher Dimensions, Superstrings and Heterotic Strings.

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Explicit result for higher loop amplitudes in superstring is quite rare. To our knowledge the only explicitly known higher loop (≥ 2) non-vanishing amplitude is the four-particle amplitude in superstring theory, firstly obtained in [1] and later re-obtained in [2, 3] in an explicitly gauge independent way, following the works of D' Hoker and Phong [4−8] on two loop measure of superstring theory. Recently D' Hoker and Phong [9, 10] also gave a measure for three loop superstring theory. It remains to see if this can be used to do explicit three loop computations in superstring theory. For other promising approach of covariant calculation of superstring amplitudes we refer the reader to Berkovits's review [11].

Due to the rareness of explicit results, it is natural to study the known result in depth. The old result was casted into an explicit modular invariant form [12] and used in [13] to prove the vanishing of the R^4 correction [14, 15]. It has also been proved in [16] that the results obtained in [1−3] are equivalent. Another goal we have in mind is to make connection with known results from field theory in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [17, 18]. This seems a trivial problem, but in fact it turns out to be quite tricky. One could use unitarity to fix the overall factor for the four-particle amplitude. In order to do this we need to know the precise overall factor for other one loop amplitudes involved. Due to the incomplete results in the literature, we therefore computed all the relevant amplitudes in a consistent way and fixed all the overall factors by either using factorization or unitarity. We will present a detailed study of factorization and unitarity in superstring theory in a later publication [19]. In this paper we present only the result of factorization of the two loop four-particle amplitude in superstring theory.

The factorization of the two loop four-particle amplitude in superstring theory was studied before in [20, 21]. Compared with these studies, we improved on obtaining some missing factors from the scalar correlators. Also some details for the calculation of the factorization of the kinematic factor are presented. We will need these results in the forthcoming paper [19].

To begin with, let us recall the two-loop four-particle amplitude in type-II superstring theories obtained in refs. [2, 3]:

$$\begin{aligned}\mathcal{A}_{II} = {} & C_{II}(2\pi)^{10}\delta^{10}\Big(\sum_i k_i\Big) g_c^4\, K(k_i,\epsilon_i) \int \frac{1}{T^5} \frac{\prod_{i=1}^6 \mathrm{d}^2 a_i}{\mathrm{d}V_{pr}|\prod_{i<j} a_{ij}|^2} \times \\ & \times \prod_{i=1}^4 \frac{\mathrm{d}^2 z_i}{|y(z_i)|^2} \prod_{i<j} \exp\{-k_i\cdot k_j \langle X(z_i)X(z_j)\rangle\} \times \\ & \times |s(z_1z_2+z_3z_4)+t(z_1z_4+z_2z_3)+u(z_1z_3+z_2z_4)|^2 \,, \end{aligned} \tag{1}$$

where

$$\mathrm{d}V_{pr} = \frac{\mathrm{d}^2a_i\mathrm{d}^2a_j\mathrm{d}^2a_k}{|a_{ij}a_{ik}a_{jk}|^2}\,, \tag{2}$$

$$T = \int \mathrm{d}^2z_1\mathrm{d}^2z_2 \frac{|z_1-z_2|^2}{|y(z_1)y(z_2)|^2}\,, \tag{3}$$

$$y^2(z) = \prod_{i=1}^6 (z-a_i)\,, \tag{4}$$

and $\langle X(z_i)X(z_j)\rangle \equiv \langle X(z_i,\bar{z}_i)X(z_j,\bar{z}_j)\rangle$'s are the scalar correlators (see below in eq. (19) for explicit formulas in terms of prime form and holomorphic differentials). The $K(k_i,\epsilon_i)$ is the standard kinematic factor appearing at tree, one- and two-loop computations [22, 1, 3]. C_{II} is an overall factor which should be determined from factorization and unitarity [19].

In the dividing degeneration limit: $a_2-a_1=u$, $a_3-a_1=vu$, $u\to 0$, we have

$$T \to \frac{2}{|u|}T_1T_2\frac{1}{|a_{14}a_{15}a_{16}|} \tag{5}$$

where

$$\begin{aligned} T_i &= \int \frac{\mathrm{d}^2 z}{|y_i(z)|^2}\,, && (6)\\ y_1(z) &= \sqrt{z(z-1)(z-v)}\,, && (7)\\ y_2(z) &= \sqrt{(z-a_1)(z-a_4)(z-a_5)(z-a_6)}\,. && (8)\end{aligned}$$

Setting

$$z_1 = x_1 u + a_1\,, \qquad \text{and} \qquad z_2 = x_2 u + a_1\,, \tag{9}$$

we have:

$$\begin{aligned} \frac{|\prod_{i<j} a_{ij}|^2}{|a_{45}a_{46}a_{56}|^2} &\to |u|^6|v(1-v)|^2|a_{14}a_{15}a_{16}|^6\,, && (10)\\ |y(z_1)|^2 &\to |u|^3|y_1(x_1)|^2|a_{14}a_{15}a_{16}|\,, && (11)\\ |y(z_4)|^2 &\to |(z_4-a_1)|^2|y_2(z_4)|^2\,, && (12)\\ \prod_{i=1}^{4}|y(z_i)|^2 &\to |u|^6|a_{14}a_{15}a_{16}|^2|(z_3-a_1)(z_4-a_1)|^2\times \\ &\quad \times|y_1(x_1)y_1(x_2)y_2(z_3)y_2(z_4)|^2\,, && (13)\\ \mathrm{d}^2a_1\mathrm{d}^2a_2\mathrm{d}^2a_3 &\to |u|^2\mathrm{d}^2u\mathrm{d}^2v\mathrm{d}^2a_1\,, && (14)\\ \mathrm{d}^2z_1\mathrm{d}^2z_2 &\to |u|^4\mathrm{d}^2x_1\mathrm{d}^2x_2\,, && (15)\\ \Big|s(z_1z_2+z_3z_4)+t(z_1z_4+z_2z_3)+u(z_1z_3+z_2z_4)\Big|^2 &\to \\ &\to |s|^2|(z_3-a_1)(z_4-a_1)|^2\,. && (16)\end{aligned}$$

By using the above results we have

$$\begin{aligned} \mathcal{A}_{II} \to\ & C_{II}(2\pi)^{10}\delta^{10}\left(\sum_i k_i\right)g_c^4K(k_i,\epsilon_i)\times \\ & \times\int\frac{\mathrm{d}^2u}{|u|2^5T_1^5T_2^5}\frac{\mathrm{d}^2a_1}{|a_{14}a_{15}a_{16}|^3}\frac{\mathrm{d}^2v}{|v(v-1)|^2}\times \\ & \times\frac{\mathrm{d}^2x_1\mathrm{d}^2x_2}{|y_1(x_1)y_1(x_2)|^2}\frac{\mathrm{d}^2z_3\mathrm{d}^2z_4}{|y_2(z_3)y_2(z_4)|^2}\times \\ & \times s^2\prod_{i<j}\exp\{-k_i\cdot k_j\langle X(z_i)X(z_j)\rangle\}\,. \end{aligned} \tag{17}$$

In order to prove that the above amplitude factorizes correctly, we need to study the degeneration limit of the scalar correlators.

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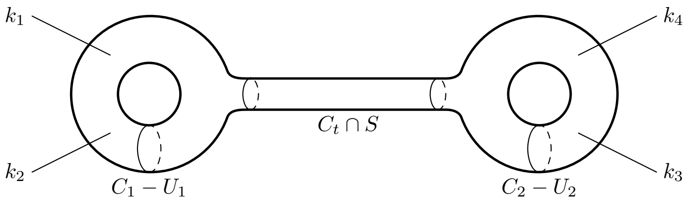

Figure 1: The dividing degeneration limit of a genus 2 surface.

The expression in the exponential factor in eq. (17) is:

$$
\begin{aligned}
-\sum_{i<j} k_i \cdot k_j \langle X(z_i) X(z_j) \rangle = \frac{s}{2}\Big(& \langle X(z_1)X(z_2)\rangle + \langle X(z_3)X(z_4)\rangle - \\
& - \langle X(z_1)X(z_3)\rangle - \langle X(z_2)X(z_4)\rangle \Big) + \\
+\frac{t}{2}\Big(& \langle X(z_1)X(z_4)\rangle + \langle X(z_2)X(z_3)\rangle - \\
& - \langle X(z_1)X(z_3)\rangle - \langle X(z_2)X(z_4)\rangle \Big) .
\end{aligned} \tag{18}
$$

where the scalar correlator $\langle X(z_i)X(z_j)\rangle$ is given as follows:

$$
\begin{aligned}
\langle X(z_i)X(z_j)\rangle = & -\ln(|E(z_i,z_j)|^2) + \\
& +2\pi \left(\operatorname{Im}\int_{z_i}^{z_j}\omega\right)(\operatorname{Im}\tau)^{-1}\left(\operatorname{Im}\int_{z_i}^{z_j}\omega\right),
\end{aligned} \tag{19}
$$

in terms of the prime form $E(z_i, z_j)$, the period matrix:

$$
\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}, \tag{20}
$$

and the holomorphic differentials:

$$
\omega(z) = \begin{pmatrix} \omega_1(z) \\ \omega_2(z) \end{pmatrix}. \tag{21}
$$

In the degeneration limit $u = t^4 \to 0$ (see figure 1), we have the following explicit formulas from Fay's book [23] (page 38, Proposition 3.1 and page 41, Corollary 3.2):

$$
\omega_1(x,t) = \begin{cases} v_1(x) + \frac{t}{4}v_1(p_1)u_1(x,p_1) + O(t^2), & (x \in C_1 - U_1) \\ \frac{t}{4}v_1(p_1)u_2(x,p_2) + O(t^2), & (x \in C_2 - U_2) \end{cases},
$$

$$
\omega_2(x,t) = \begin{cases} \frac{t}{4}v_2(p_2)u_1(x,p_1) + O(t^2), & (x \in C_1 - U_1) \\ v_2(x) + \frac{t}{4}v_2(p_2)u_2(x,p_2) + O(t^2), & (x \in C_2 - U_2) \end{cases},
$$

and

$$\tau \to \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} + 2\pi i \frac{t}{4} \begin{pmatrix} v_1^2(p_1) & v_1(p_1)v_2(p_2) \\ v_1(p_1)v_2(p_2) & v_2^2(p_2) \end{pmatrix} + O(t^2) \,. \tag{22}$$

Here $v_i(z)$ is the holomorphic differential on the i-th torus (see below in eq. (30) for explicit expression). p_1 is the ∞-point on the first torus ($p_1 = \infty$) and p_2 is one of the branch point on the second torus ($p_2 = a_1$). The abelian differential of the second kind u_i on the two tori are given as follows:

$$u_i(x,y) \;=\; -v_i(x)v_i(y)\frac{\partial^2 \ln\vartheta}{\partial^2 z}(e_i)\,, \tag{23}$$

$$e_i \;=\; \int_x^y v_i - \frac{1}{2}(\tau_i + 1)\,. \tag{24}$$

By using the above results and the degeneration property of the theta function, the correlators are degenerated as follows:

$$\langle X(z_1)X(z_2)\rangle \;\to\; -G_1(x_1,x_2) + \ln|v_1(x_1)v_1(x_2)| - \ln|u|^2\,, \tag{25}$$

$$\langle X(z_3)X(z_4)\rangle \;\to\; -G_2(z_3,z_4) + \ln|v_2(z_3)v_2(z_4)|\,, \tag{26}$$

$$\begin{aligned}\langle X(z_1)X(z_3)\rangle \;\to\; & -G_1(x_1,p_1) - G_2(p_2,z_3) + \ln|v_1(x_1)v_2(z_3)| - \\ & -\ln\left|\frac{K_1K_2}{4}\right| - \ln|u|^{1/2} - \ln|a_{14}a_{15}a_{16}|^{1/2},\end{aligned} \tag{27}$$

$$\begin{aligned}\langle X(z_2)X(z_4)\rangle \;\to\; & -G_1(x_2,p_1) - G_2(p_2,z_4) + \ln|v_1(x_2)v_2(z_4)| - \\ & -\ln\left|\frac{K_1K_2}{4}\right| - \ln|u|^{1/2} - \ln|a_{14}a_{15}a_{16}|^{1/2}\,,\end{aligned} \tag{28}$$

where we have defined:

$$G_i(x,y) = \ln\left|\frac{\vartheta_1(\int_x^y v_i,\tau_i)}{\partial\vartheta_1(0,\tau_i)}\right|^2 - 2\pi\left(\mathrm{Im}\int_x^y v_i\right)(\mathrm{Im}\,\tau_i)^{-1}\left(\mathrm{Im}\int_x^y v_i\right), \tag{29}$$

and

$$v_1(z) \;=\; \frac{\mathrm{d}x}{K_1 y_1(x)} = \mathrm{d}\tilde{x}\,, \qquad v_2(z) = \frac{\mathrm{d}z}{K_2 y_2(z)} = \mathrm{d}\tilde{z}\,, \tag{30}$$

$$K_1 \;=\; \oint_{\alpha_1}\frac{\mathrm{d}x}{y_1(x)}\,, \qquad K_2 = \oint_{\alpha_2}\frac{\mathrm{d}z}{y_2(z)}\,. \tag{31}$$

By using these results we have:

$$\begin{aligned}-\sum_{i<j} k_i\cdot k_j\langle X(z_i)X(z_j)\rangle \;\to\; \frac{s}{2}\Big[& -G_1(x_1,x_2) - G_2(z_3,z_4) + G_1(x_1,p_1) + \\ & + G_2(p_2,z_3) + G_1(x_2,p_1) + G_2(p_2,z_4) + \\ & + \ln\left(|a_{14}a_{15}a_{16}|\left|\frac{K_1K_2}{4}\right|^2\right)\Big] + \ln|u|^{-s/2}\,,\end{aligned} \tag{32}$$

and so the two loop four-boson amplitude becomes:

$$
\begin{aligned}
\mathcal{A}_{II} \to\ & iC_{II}(2\pi)^{10}\delta^{10}\Big(\sum_i k_i\Big) g_c^4\, K(k_i,\epsilon_i)\ \times \\
& \times \int \frac{\mathrm{d}^2 u}{2^5 T_1^5 T_2^5} \frac{\mathrm{d}^2 a_1}{|a_{14}a_{15}a_{16}|^3} \frac{\mathrm{d}^2 v}{|v(v-1)|^2} \times \\
& \quad \times \frac{\mathrm{d}^2 x_1 \mathrm{d}^2 x_2}{|y_1(x_1)y_1(x_2)|^2} \frac{\mathrm{d}^2 z_3 \mathrm{d}^2 z_4}{|y_2(z_3)y_2(z_4)|^2} \times \\
& \quad \times \exp\Big\{ \frac{s}{2}(-G_1(x_1,x_2) - G_2(z_3,z_4) + G_1(x_1,p_1) + \\
& \qquad\qquad + G_2(p_2,z_3) + G_1(x_2,p_1) + G_2(p_2,z_4)) \Big\} \times \\
& \quad \times s^2 |a_{14}a_{15}a_{16}|^{s/2} |K_1 K_2/4|^s |u|^{-s/2-1} \\
\to\ & iC_{II}(2\pi)^{10}\delta^{10}\Big(\sum_i k_i\Big) g_c^4\, K(k_i,\epsilon_i)\ \times \\
& \times \frac{16\pi}{-s+2} \int \frac{1}{2^5 T_1^5 T_2^5} \frac{\mathrm{d}^2 a_1}{|a_{14}a_{15}a_{16}|^2} \frac{\mathrm{d}^2 v}{|v(v-1)|^2} \times \\
& \qquad \times \frac{\mathrm{d}^2 x_1 \mathrm{d}^2 x_2}{|y_1(x_1)y_1(x_2)|^2} \frac{\mathrm{d}^2 z_3 \mathrm{d}^2 z_4}{|y_2(z_3)y_2(z_4)|^2} \times \left| \frac{K_1 K_2}{4} \right|^2 \times \\
& \qquad \times \exp\Big\{ -(G_1(x_1,x_2) - G_1(x_1,p_1) - G_1(x_2,p_1)) - \\
& \qquad\qquad - (G_2(z_3,z_4) - G_2(z_3,p_2) - G_2(z_4,p_2)) \Big\}\,.
\end{aligned} \tag{33}
$$

In order to prove that the above amplitude do factorize correctly, we need to do two things. The first is to prove that the kinematic factor $K(k_i,\epsilon_i)$ factorizes into a summation of two one-loop 3-particle kinematic factor. The summation is over all possible intermediate states. The second is to prove that the rests in eq. (33) are the product of two one-loop 3-particle amplitudes apart from the propagator $\frac{1}{-s+2}$. Let us study the second first.

The first non-vanishing contribution is from a massive intermediate states. Only the massive boson from the NS-NS sector contributes because of the super-ghost. So we need to compute the 1-loop amplitude for two massless bosons to one massive boson. The massive boson vertex operator in the 0-picture [24 – 26] is:

$$
\begin{aligned}
\mathcal{V}_M^{(0,0)}(k,\alpha_{\mu\nu\rho},\sigma_{\mu\nu},\sigma_\mu) = -g_M \Big\{ & \alpha_{[\mu\nu\rho]}[3i\partial X^\mu \psi^\nu \psi^\rho - k\cdot\psi\psi^\mu\psi^\nu\psi^\rho] + \\
& + \sigma_{\mu\nu}[i\partial X^\mu \partial X^\nu + \partial\psi^\mu\psi^\nu + k\cdot\psi i\partial X^\mu \psi^\nu] + \\
& + \sigma_\mu[i\partial^2 X + k\cdot\psi\partial\psi^\mu] \Big\} \Big\{ \text{right-moving part} \Big\}\, e^{ik\cdot X}\,,
\end{aligned} \tag{34}
$$

where $\alpha_{\mu\nu\rho}$, $\sigma_{\mu\nu}$ and σ_μ are the polarization tensors and we set the following normalization:

$$
\alpha_{\mu\nu\rho}(k)\alpha^{\mu\nu\rho}(-k) = -\frac{1}{6}\,, \qquad \sigma_{\mu\nu}(k)\sigma^{\mu\nu}(-k) = 1\,. \tag{35}
$$

By convention $\alpha(-k) = \alpha^*(k)$ and $\sigma(-k) = \sigma^*(k)$. For later use we give here also the completeness relations for these tensors:

$$\sum_\alpha \alpha_{\mu_1\nu_1\rho_1}(k)\alpha_{\mu_2\nu_2\rho_2}(-k) = -\frac{1}{288}\Big([(k_{\mu_1}k_{\mu_2} + 2g^{\mu_1\mu_2}) \times \\ \times (k_{\nu_1}k_{\nu_2} + 2g^{\nu_1\nu_2})(k_{\rho_1}k_{\rho_2} + 2g^{\rho_1\rho_2}) - \\ - (\mu_1 \leftrightarrow \nu_1) - (\mu_1 \leftrightarrow \rho_1)] - (\nu_1 \leftrightarrow \rho_1)\Big), \tag{36}$$

$$\sum_\sigma \sigma^{\mu_1\mu_2}(k)\sigma^{\nu_1\nu_2}(-k) = \frac{1}{8}\Big[(k^{\mu_1}k^{\nu_1} + 2\eta^{\mu_1\nu_1})(k^{\mu_2}k^{\nu_2} + 2\eta^{\mu_2\nu_2}) - \\ - \frac{1}{9}(k^{\mu_1}k^{\mu_2} + 2\eta^{\mu_1\mu_2})(k^{\nu_1}k^{\nu_2} + 2\eta^{\nu_1\nu_2}) + (\mu_1 \leftrightarrow \mu_2)\Big]. \tag{37}$$

The (bosonic) n-particle one loop amplitude are obtained as follows:

$$\mathcal{A}_n^{\text{1-loop}} = \int \mathrm{d}\mu \prod_{i=1}^n \mathrm{d}^2 z_i \sum_s \eta_s Q_s \Big\langle \prod_{i=1}^n \mathcal{V}(k_i, \epsilon_i, z_i)\Big\rangle_s, \tag{38}$$

and for $n = 3$ (two massless (k_1 and k_2) and one massive ($k_3 = k$)), we have

$$\begin{aligned} \mathcal{A}_3^{\text{1-loop}} &= -ig_c^2 g_M C_I (2\pi)^{10}\delta^{10}(k_1 + k_2 + k)\, K_3 \times \\ &\quad\times \int \mathrm{d}\mu\ \mathrm{d}^2 z_1\ \mathrm{d}^2 z_2\ \mathrm{d}^2 z_3 \frac{\prod_{i<j}|a_{ij}|^2}{64|y(z_1)y(z_2)y^2(z_3)|^2} \times \\ &\quad\times \exp\Big\{ -\sum_{i<j} k_i \cdot k_j \langle X(z_i)X(z_j)\rangle\Big\} \\ &= -ig_c^2 g_M C_I (2\pi)^{10}\delta^{10}(k_1 + k_2 + k)\, K_3 \times \\ &\quad\times \int \frac{1}{T^6}\frac{\prod_{i=1}^4 \mathrm{d}^2 a_i}{\mathrm{d}V_{pr}|\prod_{i<j} a_{ij}|^2}\ \frac{\mathrm{d}^2 z_1\ \mathrm{d}^2 z_2\ \mathrm{d}^2 z_3}{64\,|y(z_1)y(z_2)y^2(z_3)|^2} \times \\ &\quad\times \exp\{\langle X(z_1)X(z_2)\rangle - \langle X(z_1)X(z_3)\rangle - \langle X(z_2)X(z_3)\rangle\}, \end{aligned} \tag{39}$$

where we have used the relation $2k_i \cdot k_j = (k_i + k_j)^2 - k_i^2 - k_j^2$ to set $k_1 \cdot k_2 = -k_1 \cdot k = -k_2 \cdot k = -1$ and the kinematic factor K_3 is given as follows (for the left-part only):

$$K_{3L} = 6\alpha_{\mu\nu\rho}\epsilon_2^\mu \epsilon_1^\nu k_2^\rho + \sigma_{\mu\nu}(k_1^\mu k_2^\nu \epsilon_1 \cdot \epsilon_2 - k_1^\mu \epsilon_2^\nu \epsilon_1 \cdot k_2 - \epsilon_1^\mu \epsilon_2^\nu - \epsilon_1^\mu k_2^\nu k_1 \cdot \epsilon_2). \tag{40}$$

By comparing the one-loop amplitude given in eq. (39) with the relevant expression in eq. (33), we see that there is a mismatch of one factor of T_i and an extra integration over insertion point of the vertex operators. This is due to the fact that there is a translation invariance for the one-loop amplitude. So we can fix an insertion point and this cancels a factor T. The precise relation between G and $\langle X(z_i)X(z_j)\rangle$ is:

$$\langle X(z_i)X(z_j)\rangle = -G(z_i, z_j) - \ln|y(z_i)y(z_j)K^2|. \tag{41}$$

(See also eqs. (25)–(28).) Taking this into account, we have

$$\int \frac{1}{T^6}\frac{\mathrm{d}^2z_1\,\mathrm{d}^2z_2\,\mathrm{d}^2z}{|y(z_1)y(z_2)y^2(z)|^2}\exp\{\langle X(z_1)X(z_2)\rangle-\langle X(z_1)X(z)\rangle-\langle X(z_2)X(z)\rangle\}=$$
$$=\int\frac{|K|^2}{T^6}\frac{\mathrm{d}^2z_1\,\mathrm{d}^2z_2\,\mathrm{d}^2z}{|y(z_1)y(z_2)y(z)|^2}\exp\{-G(z_1,z_2)+G(z_1,z)+G(z_2,z)\}$$
$$=\int\frac{|K|^8}{T^6}\mathrm{d}^2\tilde{z}_1\,\mathrm{d}^2\tilde{z}_2\,\mathrm{d}^2\tilde{z}\,\exp\{-G(z_1,z_2)+G(z_1,z)+G(z_2,z)\}$$
$$=\int\frac{|K|^6}{T^5}\mathrm{d}^2\tilde{z}_1\mathrm{d}^2\tilde{z}_2\,\exp\{-G(z_1,z_2)+G(z_1,z)+G(z_2,z)\}$$
$$=\int\frac{|K|^2}{T^5}\frac{\mathrm{d}^2z_1\,\mathrm{d}^2z_2}{|y(z_1)y(z_2)|^2}\exp\{-G(z_1,z_2)+G(z_1,z)+G(z_2,z)\}\,. \tag{42}$$

This agrees with eq. (33) for the relevant part of the two one-loop amplitudes.

To complete the study of the factorization we would also like to show that the kinematic factor $K(k_i,\epsilon_i)$ also factorizes correctly. This was done in [21, 20, 27]. Here we will give more details for the summation over intermediate tensors. This factorization is true for both the left- and right-part. Setting $\epsilon_i^{\mu\nu}=\epsilon_i^{\mu}\tilde{\epsilon}_i^{\nu}$, we have

$$K(k_i,\epsilon_i)=K_L(k_i,\epsilon_i)K_R(k_i,\tilde{\epsilon}_i)\,. \tag{43}$$

In the following we discuss the factorization of K_L only and by an abuse of notation we simply write $K(k_i,\epsilon_i)$ for $K_L(k_i,\epsilon_i)$.

Setting

$$\begin{aligned}K_{3,1}(k_1,\epsilon_1,k_2,\epsilon_2,k,\alpha)&=6\epsilon_{1\mu}\epsilon_{2\nu}k_{1\rho}\alpha^{\mu\nu\rho}\\ K_{3,2}(k_1,\epsilon_1,k_2,\epsilon_2,k_3,\sigma)&=-\sigma^{\mu\nu}(\epsilon_{1\mu}\epsilon_{2\nu}+k_{1\mu}\epsilon_{2\nu}\epsilon_1\cdot k_2+\\ &\qquad+k_{1\mu}k_{1\nu}\epsilon_1\cdot\epsilon_2-\epsilon_{1\nu}k_{1\mu}\epsilon_2\cdot k_1)\,,\end{aligned} \tag{44}$$

and by using the completeness relations eqs. (36) and (37) for the polarization tensors, we have

$$\begin{aligned}&\sum_\alpha K_{3,1}(k_1,\epsilon_1,k_2,\epsilon_2,k,\alpha)K_{3,1}(k_3,\epsilon_3,k_4,\epsilon_4,-k,\alpha)=\\ &=-\frac{1}{2}\bigg\{\bigg(u\bigg(\epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3-\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4-\frac{1}{2}\epsilon_2\cdot\epsilon_3\epsilon_1\cdot k_2\epsilon_4\cdot k_3+\\ &\qquad+\frac{1}{2}\epsilon_2\cdot\epsilon_4\epsilon_1\cdot k_2\epsilon_3\cdot k_4-\frac{1}{2}\epsilon_1\cdot\epsilon_4\epsilon_2\cdot k_1\epsilon_3\cdot k_4\\ &\qquad+\frac{1}{2}\epsilon_1\cdot\epsilon_3\epsilon_2\cdot k_1\epsilon_4\cdot k_3\bigg)\bigg)+\epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3-\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4\\ &\qquad+\epsilon_2\cdot\epsilon_3(-\epsilon_1\cdot k_4\epsilon_4\cdot k_1-\epsilon_1\cdot k_3\epsilon_4\cdot k_2)\\ &\qquad-\epsilon_2\cdot\epsilon_4(-\epsilon_1\cdot k_4\epsilon_3\cdot k_1-\epsilon_1\cdot k_3\epsilon_3\cdot k_2)\\ &\qquad+\epsilon_1\cdot\epsilon_4(-\epsilon_2\cdot k_4\epsilon_3\cdot k_1-\epsilon_2\cdot k_3\epsilon_3\cdot k_2)\\ &\qquad-\epsilon_1\cdot\epsilon_3(-\epsilon_2\cdot k_4\epsilon_4\cdot k_1-\epsilon_2\cdot k_3\epsilon_4\cdot k_2)\\ &\qquad+(\epsilon_1\cdot k_4\epsilon_2\cdot k_3-\epsilon_1\cdot k_3\epsilon_2\cdot k_4)\times\\ &\qquad\times(\epsilon_3\cdot k_2\epsilon_4\cdot k_1-\epsilon_3\cdot k_1\epsilon_4\cdot k_2)\bigg\}\,,\end{aligned} \tag{45}$$

and

$$
\begin{aligned}
&\sum_\sigma K_{3,2}(k_1,\epsilon_1,k_2,\epsilon_2,k,\sigma)K_{3,2}(k_3,\epsilon_3,k_4,\epsilon_4,-k,\sigma) = \\
&\quad = \frac{1}{2}\bigg\{\frac{1}{2}u^2\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4 + u\bigg[\epsilon_3\cdot\epsilon_4(\epsilon_1\cdot k_4\epsilon_2\cdot k_3 - \epsilon_1\cdot k_3\epsilon_2\cdot k_4) + \\
&\qquad\qquad + \epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4 + \epsilon_1\cdot\epsilon_2(\epsilon_3\cdot k_2\epsilon_4\cdot k_1 - \epsilon_3\cdot k_1\epsilon_4\cdot k_2) + \\
&\qquad\qquad + \frac{1}{2}\epsilon_1\cdot\epsilon_4\epsilon_2\cdot k_1\epsilon_3\cdot k_4 + \frac{1}{2}\epsilon_2\cdot\epsilon_3\epsilon_1\cdot k_2\epsilon_4\cdot k_3 - \\
&\qquad\qquad - \frac{1}{2}\epsilon_1\cdot\epsilon_3\epsilon_2\cdot k_1\epsilon_4\cdot k_3 - \frac{1}{2}\epsilon_2\cdot\epsilon_4\epsilon_1\cdot k_2\epsilon_3\cdot k_4\bigg] + \\
&\qquad + \epsilon_2\cdot\epsilon_4(\epsilon_1\cdot k_2\epsilon_3\cdot k_1 + \epsilon_1\cdot k_3\epsilon_3\cdot k_4) + \\
&\qquad + \epsilon_1\cdot\epsilon_4(\epsilon_2\cdot k_4\epsilon_3\cdot k_1 - \epsilon_2\cdot k_3\epsilon_3\cdot k_2) + \\
&\qquad + \epsilon_2\cdot\epsilon_3(\epsilon_1\cdot k_3\epsilon_4\cdot k_2 - \epsilon_1\cdot k_4\epsilon_4\cdot k_1) + \\
&\qquad + \epsilon_1\cdot\epsilon_3(\epsilon_2\cdot k_1\epsilon_4\cdot k_2 + \epsilon_2\cdot k_4\epsilon_4\cdot k_3) - \\
&\qquad - 2\epsilon_1\cdot\epsilon_2\epsilon_3\cdot k_1\epsilon_4\cdot k_2 - 2\epsilon_3\cdot\epsilon_4\epsilon_1\cdot k_3\epsilon_2\cdot k_4 + \\
&\qquad + \epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3 + \epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 + \\
&\qquad + (\epsilon_1\cdot k_4\epsilon_2\cdot k_3 - \epsilon_1\cdot k_3\epsilon_2\cdot k_4)\times \\
&\qquad \times(\epsilon_3\cdot k_2\epsilon_4\cdot k_1 - \epsilon_3\cdot k_1\epsilon_4\cdot k_2)\bigg\}\,.
\end{aligned}
\tag{46}
$$

By using the above results one proves that the kinematic factor $K(k_i,\epsilon_i)$ factorizes into a product of two one-loop kinematic factors. Explicitly we have:

$$
\begin{aligned}
K(k_i,\epsilon_i)|_{(k_1+k_2)^2=-2} &= \frac{u^2}{4}\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4 + \\
&\quad + \frac{u}{2}\Big[\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4 + \epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 - \\
&\qquad - \epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3 + \epsilon_1\cdot\epsilon_2(\epsilon_3\cdot k_2\epsilon_4\cdot k_1 - \epsilon_3\cdot k_1\epsilon_4\cdot k_2) + \\
&\qquad + \epsilon_3\cdot\epsilon_4(\epsilon_1\cdot k_4\epsilon_2\cdot k_3 - \epsilon_1\cdot k_3\epsilon_2\cdot k_4) + \\
&\qquad + \epsilon_1\cdot\epsilon_4\epsilon_2\cdot k_1\epsilon_3\cdot k_4 + \epsilon_2\cdot\epsilon_3\epsilon_1\cdot k_2\epsilon_4\cdot k_3 - \\
&\qquad - \epsilon_1\cdot\epsilon_3\epsilon_2\cdot k_1\epsilon_4\cdot k_3 - \epsilon_2\cdot\epsilon_4\epsilon_1\cdot k_2\epsilon_3\cdot k_4\Big] + \\
&\quad + \epsilon_1\cdot\epsilon_3(\epsilon_2\cdot k_3\epsilon_4\cdot k_1 - \epsilon_2\cdot k_1\epsilon_4\cdot k_3) + \\
&\quad + \epsilon_2\cdot\epsilon_4(\epsilon_1\cdot k_4\epsilon_3\cdot k_2 - \epsilon_1\cdot k_2\epsilon_3\cdot k_4) + \\
&\quad + \epsilon_2\cdot\epsilon_3\epsilon_1\cdot k_3\epsilon_4\cdot k_2 + \epsilon_1\cdot\epsilon_4\epsilon_2\cdot k_4\epsilon_3\cdot k_1 - \epsilon_1\cdot\epsilon_2\epsilon_3\cdot k_1\epsilon_4\cdot k_2 - \\
&\quad - \epsilon_3\cdot\epsilon_4\epsilon_2\cdot k_4\epsilon_1\cdot k_3 + \epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 \\
&= \sum_\alpha K_{3,1}(k_1,\epsilon_1,k_2,\epsilon_2,k,\alpha)K_{3,1}(k_3,\epsilon_3,k_4,\epsilon_4,-k,\alpha) + \\
&\quad + \sum_\sigma K_{3,2}(k_1,\epsilon_1,k_2,\epsilon_2,k,\sigma)K_{3,2}(k_3,\epsilon_3,k_4,\epsilon_4,-k,\sigma)\,.
\end{aligned}
\tag{47}
$$

with $k=-(k_1+k_2)$. This completes the study of the factorization of the two-loop four-particle amplitude in superstring theory.

One could use the above result to fix the overall factor of the two-loop four-particle amplitude. In order to do this we need the precise overall factor for the one-loop three-particle amplitude. This factor is unknown in the literature and it should be determined by either using unitarity or factorization at one-loop. We will leave these for a future publication [19].

Note added: recently the 2-loop 4-particle amplitude in superstring theory was also obtained in [28, 29] and its factorization was studied by D'Hoker, Gutperle and Phong in [30].

References

[1] R. Iengo and C.-J. Zhu, *Two loop computation of the four particle amplitude in heterotic string theory*, *Phys. Lett.* **B 212** (1988) 313.

[2] Z.-J. Zheng, J.-B. Wu and C.-J. Zhu, *Two-loop superstrings in hyperelliptic language, I. The main results*, *Phys. Lett.* **B 559** (2003) 89 [hep-th/0212191].

[3] Z.-J. Zheng, J.-B. Wu and C.-J. Zhu, *Two-loop superstrings in hyperelliptic language, III. The four-particle amplitude*, *Nucl. Phys.* **B 663** (2003) 95 [hep-th/0212219].

[4] E. D'Hoker and D.H. Phong, *Two-loop superstrings, I. Main formulas*, *Phys. Lett.* **B 529** (2002) 241 [hep-th/0110247].

[5] E. D'Hoker and D.H. Phong, *Two-loop superstrings, II. The chiral measure on moduli space*, *Nucl. Phys.* **B 636** (2002) 3 [hep-th/0110283].

[6] E. D'Hoker and D.H. Phong, *Two-loop superstrings III, slice independence and absence of ambiguities*, *Nucl. Phys.* **B 636** (2002) 61 [hep-th/0111016].

[7] E. D'Hoker and D.H. Phong, *Two-loop superstrings, IV. The cosmological constant and modular forms*, *Nucl. Phys.* **B 639** (2002) 129 [hep-th/0111040].

[8] For reviews, see E. D'Hoker and D.H. Phong, *Lectures on two-loop superstrings*, hep-th/0211111; *The geometry of string perturbation theory*, *Rev. Mod. Phys.* **60** (1988) 917.

[9] E. D'Hoker and D.H. Phong, *Asyzygies, modular forms and the superstring measure, I*, *Nucl. Phys.* **B 710** (2005) 58 [hep-th/0411159].

[10] E. D'Hoker and D.H. Phong, *Asyzygies, modular forms and the superstring measure, II*, *Nucl. Phys.* **B 710** (2005) 83 [hep-th/0411182].

[11] N. Berkovits, *Covariant multiloop superstring amplitudes*, *Comptes Rendus Physique* **6** (2005) 185 [hep-th/0410079].

[12] R. Iengo and C.-J. Zhu, *Explicit modular invariant two-loop superstring amplitude relevant for R^4*, *JHEP* **06** (1999) 011 [hep-th/9905050].

[13] R. Iengo, *Computing the R^4 term at two super-string loops*, *JHEP* **02** (2002) 035 [hep-th/0202058].

[14] D.J. Gross and E. Witten, *Superstring modifications of Einstein's equations*, *Nucl. Phys.* **B 277** (1986) 1.

[15] M.B. Green and M. Gutperle, *Effects of d-instantons*, *Nucl. Phys.* **B 498** (1997) 195 [hep-th/9701093].

JHEP06(2005)002

[16] W.-J. Bao and C.-J. Zhu, *Comments on two-loop four-particle amplitude in superstring theory*, *JHEP* **05** (2003) 056 [hep-th/0303152].

[17] Z. Bern, J.S. Rozowsky and B. Yan, *Two-loop four-gluon amplitudes in $N = 4$ super–Yang-Mills*, *Phys. Lett.* **B 401** (1997) 273 [hep-ph/9702424].

[18] Z. Bern, *Perturbative quantum gravity and its relation to gauge theory*, *Living Rev. Rel.* **5** (2002) 5 [gr-qc/0206071].

[19] Z.-G. Xiao and C.-J. Zhu, *Factorization and unitarity in superstring theory*, hep-th/0503248.

[20] C.-J. Zhu, *Factorization of a two loop four particle amplitude in superstring theories*, *Phys. Lett.* **B 221** (1989) 27.

[21] O. Yasuda, *Factorization of a two loop four point superstring amplitude*, *Phys. Rev. Lett.* **60** (1988) 1688.

[22] M.B. Green and J.H. Schwarz, *Supersymmetrical dual string theory, 3. Loops and renormalization*, *Nucl. Phys.* **B 198** (1982) 441.

[23] J. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Mathematics, **352**, Springer-Verlag, Berlin 1973.

[24] G. Aldazabel, M. Bonini, R. Jengo and C. Núñez, *Superstring vertex operators and scattering amplitudes on arbitrary Riemann surfaces*, *Nucl. Phys.* **B 307** (1988) 291.

[25] I.G. Koh, W. Troost and A. Van Proeyen, *Covariant higher spin vertex operators in the Ramond sector*, *Nucl. Phys.* **B 292** (1987) 201.

[26] Y. Tanii and Y. Watabiki, *Vertex functions in the path integral formalism of string theories*, *Nucl. Phys.* **B 316** (1989) 171.

[27] C.-J. Zhu, *Superstring perturbation theory up to two loops and beyond*, SISSA Ph.D. thesis 1989, unpublished.

[28] E. D'Hoker and D.H. Phong, *Two-loop superstrings, V. Gauge slice independence of the N-point function*, *Nucl. Phys.* **B 715** (2005) 91 [hep-th/0501196].

[29] E. D'Hoker and D.H. Phong, *Two-loop superstrings, VI. Non-renormalization theorems and the 4-point function*, *Nucl. Phys.* **B 715** (2005) 3 [hep-th/0501197].

[30] E. D'Hoker, M. Gutperle and D.H. Phong, *Two-loop superstrings and S-duality*, hep-th/0503180.